

## Path Integration on Hermitian Hyperbolic Space

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### Abstract

In this paper the path integral technique is applied to the quantum motion on the Hermitian hyperbolic space  $HH(2)$ . The Schrödinger equation on this space separates in 12 coordinate systems which are closely related to the coordinate systems on the two-dimensional hyperboloid. For six coordinate systems out of the twelve it is possible to find a path integral solution.

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## 1 Introduction

In the present paper the path integral method [7, 21, 32, 42] is applied to the Hermitian hyperbolic space  $\text{HH}(2)$ . This work is the continuation of the program to apply the path integral formalism to as many as possible quantum systems. In recent publications we have achieved path integral solutions of two- and three-dimensional flat space  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , on the two- and three-dimensional sphere  $S^{(2)}$  and  $S^{(3)}$ , and the two- and three-dimensional hyperboloid  $\Lambda^{(2)}$  and  $\Lambda^{(3)}$  [14, 15]. Also some other specific cases were considered, like imaginary Lobachevsky Space [13] or hyperbolic spaces of rank one [11]. Whereas in some of these manifolds just spherical coordinates, or coordinates related to them, were used to evaluate the path integral, a systematic study was performed for spaces in two- and three dimensions with constant (zero, positive, or negative) curvature, i.e., Cartesian space, spheres and hyperboloids. As a general observation, it was possible to solve the path integral explicitly in coordinate systems which were *non-parametric*, e.g. spherical or parabolic coordinates. Parametric coordinate systems were more difficult to handle. Important examples of the solution of the path integral in a parametric coordinate system are elliptic and spheroidal coordinates in flat space [15, 21] and on spheres [16]. In these cases a theory of special functions, the elliptic and spheroidal functions, exists [36]. Some of these results could be applied by a heuristic analytic continuation to the three-dimensional hyperboloid.

These results were summarised in the monography [15]. In our “Handbook of Feynman Path Integrals” [21] we collected as best to our knowledge all known solutions for the Feynman path integral in quantum mechanics. Here also many references were collected and we rely on this in the sequel, if a known path integral solution, say for a potential problem, must be applied in a subsequent path integration in a particular coordinate system in a hyperbolic space.

It is worth noting that the Basic Path Integrals, by which we mean the path integral solution of the (radial) harmonic oscillator, the (modified) Pöschl–Teller potential, and the spheroidal path integral, respectively, were found by means of a group-space path integration. Of particular importance are the two cases of the Pöschl–Teller potential [1, 6, 8, 33] (SU(2)-group path integration) and the modified Pöschl–Teller potential [1, 8, 33] (SU(1, 1)-group path integration). This has now been generally established in the literature, and will not be repeated here in much detail.

In the last years many textbooks have been published which were devoted to the application of the path integral method in various branches of mathematical physics, e.g. by Haba [22], Johnson and Lapidus [25], Kolokol'tsov [34], and many others as has been listed in our publication [21]. Two further important publications are due to Inomata, Kuratsuji and Gery [24], and Tomé [43], where path integrals and coherent states based on SU(2) and SU(1, 1) were discussed, together with applications to potential problems.

Let us shortly discuss the physical significance of the consideration of separation of variables in coordinate systems. The free motion in some space is, of course, the most symmetric one, and the search for the number of coordinate systems which allow the separation of the Hamiltonian is equivalent to the investigation how many inequivalent sets of variables can be found. The incorporation of potentials usually removes at least some of the symmetry properties of the space. Well-known examples are spherical systems, and they are most conveniently studied in spherical coordinates. For instance, the isotropic harmonic oscillator in three dimensions is separable in eight coordinate systems, namely in Cartesian, spherical, circular polar, circular elliptic, conical, oblate spheroidal, prolate spheroidal, and ellipsoidal coordinates. The Coulomb potential is separable in four coordinate systems, namely in conical, spherical parabolic, and prolate spheroidal II coordinates.

The separation of a particular quantum mechanical potential problem into more than one coordinate system has the consequence that there are additional integrals of motion and that the spectrum is degenerate. The Noether theorem connects the particular symmetries of a Lagrangian, i.e., the invariances with respect to the dynamical symmetries, with conservation laws in classical mechanics and with observables in quantum mechanics, respectively. In the case of the isotropic harmonic oscillator one has in addition to the conservation of energy and the conservation of the angular momentum, the conservation of the quadrupole moment; in the case of the Coulomb problem one has in addition to the conservation of energy and the angular momentum, the conservation of the Pauli-Runge-Lenz vector. In total, the additional conserved quantities in these two examples add up to five functionally independent integrals of motion in classical mechanics, respectively observables in quantum mechanics.

Disturbing the symmetry usually spoils it. This can be achieved by adding terms into the Hamiltonian which are non-symmetric. Maximally super-integrable systems turn into minimally superintegrable systems, into just integrable, or non-integrable systems. The integrable systems may not be explicitly solvable but may remain separable. A well known example is the two-Coulomb centre problem which is separable in spheroidal coordinates, but is not explicitly solvable in terms of known higher-transcendental functions.

Another motivation for studying a system in terms of separation in different coordinates systems is the property of coordinate systems that they represent different physical set-ups in scattering theory, e.g. Wehrhahn et al. [47].

The comprehensive results of the evaluation of the path integral in spaces of two and three-dimensions were possible because the number and form of the coordinate systems which allow separation of variables in the Helmholtz, respectively the Schrödinger equation, and therefore also for the path integral, are known. For the cases of flat (real or complex) spaces, spheres and hyperboloids this is known for a long time, e.g. [31, 37, 38, 39]. However, for other spaces this is in general not the case. A method how to construct and find coordinate systems on homogeneous spaces is known and has been applied for Minkowski spaces [26] and higher dimensional hyperbolic spaces [27]. Some of the corresponding path integrals evaluations were presented in [15]. A particular feature of the path integral solution (i.e. the integral kernel of the time-evolution operator) on spheres, flat space, and hyperboloids was, that the corresponding Green's function (i.e. the integral kernel of the resolvent operator) could be expressed in closed form: In flat space one obtains for the principal term a  $K$ -Bessel function (which in odd dimensions can be simplified to an exponentials times a power-term), on spheres one obtains a Legendre-function  $P_\nu^\mu(x)$  of the first kind (which in odd dimensions can be simplified to powers of trigonometric functions), and on hyperboloids one obtains a Legendre-function  $Q_\nu^\mu(z)$  of the second kind. In all these cases the Green's function depends only on the invariant distance  $d$  in the space in question. In flat space, this is the Euclidean distance  $d = d(|\mathbf{x} - \mathbf{y}|)$  ( $\mathbf{x}, \mathbf{y} \in \mathbb{R}^D$ ), on the sphere it is the angle  $\psi(\{\vartheta\})$  ( $\{\vartheta\}$  spherical angles), and on hyperboloids it is the hyperbolic distance  $d(\mathbf{u}'', \mathbf{u}')$  ( $\mathbf{u}$  element of the hyperboloid). In more general cases of hyperbolic spaces, group theoretic tools can be used to derive integral representations [46].

The Hermitian hyperbolic space  $\text{HH}(n)$  is defined by  $\text{SU}(n, 1)/\text{S}[\text{U}(1) \times \text{U}(n)]$  (see e.g. Helgason [23] or Venkov [45]).  $\text{SU}(n, 1)$  is the isometry group of  $\text{HH}(n)$  that leaves the Hermitian form invariant, and  $\text{S}[\text{U}(1) \times \text{U}(n)] = \text{SU}(n, 1) [\text{U}(1) \times \text{U}(n)]$  is an isotropy subgroup of the isometry group. For  $\text{HH}(2)$ , Boyer et al. [2] found twelve coordinate systems which allow separation of variables in the Helmholtz, respectively the Schrödinger equation, and the path integral. In [2], for example, mutually non-conjugate maximal Abelian subgroups of  $\text{SU}(2, 1)$  are used to construct separable coordinate systems. The special feature of the isotropy group is that it has four mutually non-conjugate maximal Abelian subgroups, which give rise to the fact that each of the separable coordinate systems has *exactly* two so-called ignorable coordinates [3]. Ignorable coordinates do not appear in the metric tensor explicitly, and in the corresponding quantum Hamiltonian they just give two-fold partial differentials, therefore they giving simple plane-waves or circular-waves as solutions of the Hamiltonian. The remaining two coordinates can be classified by means of the nine coordinate systems on the two-dimensional hyperboloid. Combining properly the sub-algebras yields twelve coordinate systems on  $\text{HH}(2)$  [2]. This will not be repeated here.

The present system is of interest due to the structure of the metric which has the form  $(-, +, \dots, +)$ , i.e. it is of the Minkowski-type, and the Hamiltonian system under consideration is integrable and relativistic with non-trivial interaction after integrating out the ignorable variables [2]. This feature of constructing interaction, respectively potential forces, is also known from examples of quantum motion on other group spaces [1, 6, 8, 33].

I do not want to go into the details of the construction of the Hermitian hyperbolic space  $\text{HH}(n)$  in general and for  $\text{HH}(2)$  in particular. Detailed information can be found in [2] The

Hamiltonian for HH(2) has the form

$$\mathcal{H} = \frac{4}{2m}(1 - |z_1|^2 - |z_2|^2) \left[ (|z_1|^2 - 1)|p_{z_1}|^2 + (|z_2|^2 - 1)|p_{z_2}|^2 + z_1 \bar{z}_2 p_{z_1} \bar{p}_{z_2} + \bar{z}_1 z_2 \bar{p}_{z_1} p_{z_2} \right] . \quad (1.1)$$

and this information will be sufficient for our purposes.

In the following we present the twelve coordinate systems. As we will see, in six out of the twelve systems we can explicitly evaluate the path integral. We cannot find a path integral solution of the three parametric systems and the three parabolic systems. We find path integral solutions for the spherical, the three equidistant, and the two horicyclic coordinate systems. After the statement of the coordinate systems, the Hamiltonian is given (following [2]) and then the metric tensor is extracted. Of course, well know path integral solutions come into play. The ignorable coordinates can be separated off in the path integrals by a two-dimensional Gaussian path integration. For the convenience of the reader, I briefly sketch the path integral definition which is used in this paper. An exact lattice definition of a path integral in a curved space is important, because different lattice definitions and their corresponding different ordering prescriptions in the quantum Hamiltonian must not be mixed up. In the Conclusions the results are summarised and discussed.

## 2 The Path Integral Solutions The Spherical Coordinate System

The spherical coordinate system on HH(2) is given by

$$\left. \begin{aligned} z_1 &= \tanh \omega \cos \beta e^{i\varphi_1} \\ z_2 &= \tanh \omega \sin \beta e^{i\varphi_2} \end{aligned} \right\} \quad (\omega > 0, \beta \in (0, \frac{\pi}{2}), \varphi_1, \varphi_2 \in [0, 2\pi)) . \quad (2.1)$$

This gives for the Hamiltonian

$$\mathcal{H}(\omega, p_\omega, \beta, p_\beta, p_{\varphi_1}, p_{\varphi_2}) = \frac{1}{2m} \left[ p_\omega^2 + \frac{1}{\sinh^2 \omega} \left( p_\beta^2 + \frac{p_{\varphi_1}^2}{\cos^2 \beta} + \frac{p_{\varphi_2}^2}{\sin^2 \beta} \right) + \frac{(p_{\varphi_1} + p_{\varphi_2})^2}{\cosh^2 \omega} \right] \quad (2.2)$$

$$= \frac{1}{2m} \left[ p_\omega^2 + \frac{p_\beta^2}{\sinh^2 \omega} + A(\omega, \beta) p_{\varphi_1}^2 + B(\omega, \beta) p_{\varphi_2}^2 + \frac{2p_{\varphi_1} p_{\varphi_2}}{\cosh^2 \omega} \right] , \quad (2.3)$$

with the quantities  $A(\omega, \beta)$  and  $B(\omega, \beta)$  given by

$$A(\omega, \beta) = \frac{1}{\sinh^2 \omega \cos^2 \beta} - \frac{1}{\cosh^2 \omega} , \quad B(\omega, \beta) = \frac{1}{\sinh^2 \omega \sin^2 \beta} - \frac{1}{\cosh^2 \omega} . \quad (2.4)$$

Therefore we obtain for the (inverse) metric tensor ( $g^{ab}$ ):

$$(g^{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sinh^2 \omega} & 0 & 0 \\ 0 & 0 & A(\omega, \beta) & -\frac{1}{\cosh^2 \omega} \\ 0 & 0 & -\frac{1}{\cosh^2 \omega} & B(\omega, \beta) \end{pmatrix} \quad (2.5)$$

which gives

$$\sqrt{g} = \sqrt{\det(g_{ab})} = \sinh^3 \omega \cosh \omega \sin \beta \cos \beta . \quad (2.6)$$

Let us abbreviate

$$(\hat{g}^{ab}) = \begin{pmatrix} A(\omega, \beta) & -\frac{1}{\cosh^2 \omega} \\ -\frac{1}{\cosh^2 \omega} & B(\omega, \beta) \end{pmatrix} , \quad (2.7)$$

then it follows

$$(\hat{g}_{ab}) = \begin{pmatrix} \sinh^2 \omega \cos^2 \beta (\cosh^2 \omega - \sinh^2 \omega \sin^2 \beta) & \sinh^4 \omega \sin^2 \beta \cos^2 \beta \\ \sinh^4 \omega \sin^2 \beta \cos^2 \beta & \sinh^2 \omega \sin^2 \beta (\cosh^2 \omega - \sinh^2 \omega \cos^2 \beta) \end{pmatrix} . \quad (2.8)$$

Therefore we can write the Lagrangian in the following from

$$\begin{aligned} \mathcal{L} = \frac{m}{2} & \left\{ \dot{\omega}^2 + \sinh^2 \omega \dot{\beta}^2 + \sinh^2 \omega \left[ \cosh^2 \omega (\cos^2 \beta \dot{\varphi}_1^2 + \sin^2 \beta \dot{\varphi}_2^2) \right. \right. \\ & \left. \left. - \sinh^2 \omega \sin^2 \beta \cos^2 \beta (\dot{\varphi}_1 - \dot{\varphi}_2)^2 \right] \right\} \quad (2.9) \end{aligned}$$

$$= \frac{m}{2} \left[ \dot{\omega}^2 + \sinh^2 \omega \dot{\beta}^2 + (\dot{\varphi}_1, \dot{\varphi}_2) (\hat{g}_{ab}) \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} \right] \quad (2.10)$$

From these ingredients we find for the momentum operators

$$p_\omega = \frac{\hbar}{i} \left( \frac{\partial}{\partial \omega} + \frac{3}{2} \coth \omega + \frac{1}{2} \tanh \omega \right) , \quad (2.11)$$

$$p_\beta = \frac{\hbar}{i} \left( \frac{\partial}{\partial \beta} + \frac{1}{2} (\cot \beta - \tan \beta) \right) , \quad (2.12)$$

$$p_{\varphi_1} = \frac{\hbar}{i} \frac{\partial}{\partial \varphi_1} , \quad p_{\varphi_2} = \frac{\hbar}{i} \frac{\partial}{\partial \varphi_2} . \quad (2.13)$$

The quantum potential according to our ordering prescription is found to read

$$\Delta V(\omega, \beta) = -\frac{\hbar^2}{8m} \left[ \left( \frac{1}{\sinh^2 \omega} - \frac{1}{\cosh^2 \omega} - 16 \right) + \frac{1}{\sinh^2 \omega} \left( \frac{1}{\sin^2 \beta} + \frac{1}{\cos^2 \beta} \right) \right] . \quad (2.14)$$

Starting from Eq.(2.2) we have extracted the corresponding metric tensor, therefore got also its inverse, and found the corresponding Lagrangian. In the path integral formalism this procedure corresponds from starting with the Hamiltonian path integral, and by integrating out the (Gaussian) momentum-path-integrations obtaining the Lagrangian path integral. This is always possible provided the Hamiltonian, respectively the Lagrangian, are not singular. It is in effect also the canonical method to construct the path Lagrangian integral by starting with a proper *Hamiltonian operator* and its corresponding classical *Hamiltonian function*. In this correspondence we have to take into account a proper ordering prescription of momentum and position operators in the Hamiltonian operator. However, this is a well-defined prescription which has been extensively worked out in [21], where also a detailed overview of several ordering prescriptions and their differences, advantages and disadvantages was given.

We have all the ingredients to our disposal to set up the path integral in spherical coordinates on HH(2). We obtain

$$\begin{aligned}
& K(\omega'', \omega', \beta'', \beta', \varphi_1'', \varphi_1', \varphi_2'', \varphi_2'; T) \\
&= \int_{\omega(t')=\omega'}^{\omega(t'')=\omega''} \mathcal{D}\omega(t) \int_{\beta(t')=\beta'}^{\beta(t'')=\beta''} \mathcal{D}\beta(t) \int_{\varphi_1(t')=\varphi_1'}^{\varphi_1(t'')=\varphi_1''} \mathcal{D}\varphi_1(t) \int_{\varphi_2(t')=\varphi_2'}^{\varphi_2(t'')=\varphi_2''} \mathcal{D}\varphi_2(t) \sinh^3 \omega \cosh \omega \sin \beta \cos \beta \\
&\quad \times \exp \left( \frac{i}{\hbar} \int_0^T \left\{ \frac{m}{2} \left[ \dot{\omega}^2 + \sinh^2 \omega \dot{\beta}^2 + (\dot{\varphi}_1, \dot{\varphi}_2)(\hat{g}_{ab}) \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} \right] \right. \right. \\
&\quad \left. \left. + \frac{\hbar^2}{8m} \left[ \left( \frac{1}{\sinh^2 \omega} - \frac{1}{\cosh^2 \omega} - 16 \right) + \frac{1}{\sinh^2 \omega} \left( \frac{1}{\sin^2 \beta} + \frac{1}{\cos^2 \beta} \right) \right] \right\} dt \right) . \quad (2.15)
\end{aligned}$$

This path integral is evaluated in the first step by means of a Fourier expansion according to

$$K_{k_1 k_2}(\omega'', \omega', \beta'', \beta'; T) = \int d\varphi_1 d\varphi_2 e^{-i(k_1 \varphi_1 + k_2 \varphi_2)} K(\omega'', \omega', \beta'', \beta', \varphi_1'', \varphi_1', \varphi_2'', \varphi_2'; T) \quad (2.16)$$

$$K(\omega'', \omega', \beta'', \beta', \varphi_1'', \varphi_1', \varphi_2'', \varphi_2'; T) = \sum_{k_1, k_2 \in \mathbb{Z}^2} \frac{e^{ik_1(\varphi_1'' - \varphi_1')}}{2\pi} \frac{e^{ik_2(\varphi_2'' - \varphi_2')}}{2\pi} K_{k_1 k_2}(\omega'', \omega', \beta'', \beta'; T). \quad (2.17)$$

We make use of the general Gaussian integral (in  $D$  dimensions)

$$\int d\mathbf{p} e^{i\mathbf{q} \cdot \mathbf{p} - \frac{1}{2} g^{ab} p_a p_b} = (2\pi)^{D/2} \sqrt{\det(g_{ab})} e^{-\frac{1}{2} g_{ab} q^a q^b}. \quad (2.18)$$

We see that we can separate the  $(\varphi_1, \varphi_2)$ -coordinates in the Lagrangian path integral. The corresponding quantum numbers  $(k_1, k_2)$  yield via (2.18), respectively with  $g^{ab}$  replaced by  $g_{ab}$ , potential terms reflecting the corresponding terms in the Hamiltonian  $\mathcal{H}$  (2.2) where the momenta  $(p_{\varphi_1}, p_{\varphi_2})$  are replaced by  $(-i\hbar k_1, -i\hbar k_2)$ . We obtain (by displaying explicitly the lattice definition in  $(\varphi_1, \varphi_2)$ )

$$\begin{aligned}
& K_{k_1 k_2}(\omega'', \omega', \beta'', \beta'; T) \\
&= \int_{\omega(t')=\omega'}^{\omega(t'')=\omega''} \mathcal{D}\omega(t) \int_{\beta(t')=\beta'}^{\beta(t'')=\beta''} \mathcal{D}\beta(t) \sqrt{g} \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} (\dot{\omega}^2 + \sinh^2 \omega \dot{\beta}^2) - \Delta V(\omega, \beta) \right] dt \right\} \\
&\quad \times \prod_{j=1}^N \frac{m}{2\pi i \epsilon \hbar} \underbrace{\int d\varphi_{1,j} \int d\varphi_{2,j} e^{-\frac{m}{2i\epsilon\hbar} (\Delta\varphi_{1,j}, \Delta\varphi_{2,j}) (\hat{g}_{ab}(\omega_j, \beta_j)) \begin{pmatrix} \Delta\varphi_{1,j} \\ \Delta\varphi_{2,j} \end{pmatrix} - i k_1 \Delta\varphi_{1,j} - i k_2 \Delta\varphi_{2,j}}_{= \frac{2\pi i \epsilon \hbar}{m} \sqrt{\det(\hat{g}^{ab})} \exp \left[ -\frac{i\epsilon\hbar}{2m} (k_1, k_2) (\hat{g}^{ab}) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \right]} \\
&= (\sinh^2 \omega'' \sinh^2 \omega' \cosh \omega'' \cosh \omega' \sin \beta'' \sin \beta' \cos \beta'' \cos \beta')^{-1/2} e^{-2i\hbar T/m} \\
&\quad \times \int_{\omega(t')=\omega'}^{\omega(t'')=\omega''} \mathcal{D}\omega(t) \int_{\beta(t')=\beta'}^{\beta(t'')=\beta''} \mathcal{D}\beta(t) \sinh \omega \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} (\dot{\omega}^2 + \sinh^2 \omega \dot{\beta}^2) \right. \right. \\
&\quad \left. \left. - \frac{\hbar^2}{2m \sinh^2 \omega} \left( \frac{k_1^2 - \frac{1}{4}}{\cos^2 \beta} + \frac{k_2^2 - \frac{1}{4}}{\sin^2 \beta} - \frac{1}{4} \right) + \frac{\hbar^2}{2m} \frac{(k_1 + k_2)^2 - \frac{1}{4}}{\cosh^2 \omega} \right] dt \right\}. \quad (2.19)
\end{aligned}$$

The above path integral is first in the variable  $\beta$  a path integral for the Pöschl–Teller potential with a discrete spectrum and quantum number  $n$ , and second in the variable  $\omega$  a path integral for the modified Pöschl–Teller potential with a continuous spectrum and the quantum number  $p$ . Therefore we can write down the complete solution as follows

$$K(\omega'', \omega', \beta'', \beta', \varphi_1'', \varphi_1', \varphi_2'', \varphi_2'; T) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \int_0^\infty dp \Psi_{p,n,k_1 k_2}(\omega'', \beta'', \varphi_1'', \varphi_2'') \Psi_{p,n,k_1 k_2}^*(\omega', \beta', \varphi_1', \varphi_2') e^{-iE_p T/\hbar} , \quad (2.20)$$

with the wave-functions and the energy-spectrum given by

$$\Psi_{p,n,k_1 k_2}(\omega, \beta, \varphi_1, \varphi_2) = \left(\frac{1}{4} \sinh 2\omega \sin 2\beta\right)^{-1/2} \frac{e^{i(k_1 \varphi_1 + k_2 \varphi_2)}}{2\pi} \Phi_n^{(k_1, k_2)}(\beta) \Psi_p^{(k_1 + k_2 + 2n - 1, k_1 + k_2)}(\omega) , \quad (2.21)$$

$$E_p = \frac{\hbar^2}{2m}(p^2 + 4) . \quad (2.22)$$

The  $\Phi_n^{(k_1, k_2)}(\beta)$  are the Pöschl–Teller functions, which are given by [1, 6, 8, 33]

$$V(x) = \frac{\hbar^2}{2m} \left( \frac{\alpha^2 - \frac{1}{4}}{\sin^2 x} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 x} \right) \Phi_n^{(\alpha, \beta)}(x) \quad (2.23)$$

$$= \left[ 2(\alpha + \beta + 2l + 1) \frac{l! \Gamma(\alpha + \beta + l + 1)}{\Gamma(\alpha + l + 1) \Gamma(\beta + l + 1)} \right]^{1/2} \times (\sin x)^{\alpha+1/2} (\cos x)^{\beta+1/2} P_n^{(\alpha, \beta)}(\cos 2x) . \quad (2.24)$$

The  $P_n^{(\alpha, \beta)}(z)$  are Gegenbauer polynomials. The  $\Psi_p^{(\mu, \nu)}(\omega)$  are the modified Pöschl–Teller functions, which are given by [1, 6, 8, 33]

$$\Psi_n^{(\eta, \nu)}(r) = N_n^{(\eta, \nu)} (\sinh r)^{2k_2 - \frac{1}{2}} (\cosh r)^{-2k_1 + \frac{3}{2}} \times {}_2F_1(-k_1 + k_2 + \kappa, -k_1 + k_2 - \kappa + 1; 2k_2; -\sinh^2 r) \quad (2.25)$$

$$N_n^{(\eta, \nu)} = \frac{1}{\Gamma(2k_2)} \left[ \frac{2(2\kappa - 1) \Gamma(k_1 + k_2 - \kappa) \Gamma(k_1 + k_2 + \kappa - 1)}{\Gamma(k_1 - k_2 + \kappa) \Gamma(k_1 - k_2 - \kappa + 1)} \right]^{1/2} . \quad (2.26)$$

The scattering states are given by:

$$V(r) = \frac{\hbar^2}{2m} \left( \frac{\eta^2 - \frac{1}{4}}{\sinh^2 r} - \frac{\nu^2 - \frac{1}{4}}{\cosh^2 r} \right) \Psi_p^{(\eta, \nu)}(r) = N_p^{(\eta, \nu)} (\cosh r)^{2k_1 - \frac{1}{2}} (\sinh r)^{2k_2 - \frac{1}{2}} \times {}_2F_1(k_1 + k_2 - \kappa, k_1 + k_2 + \kappa - 1; 2k_2; -\sinh^2 r) \quad (2.27)$$

$$N_p^{(\eta, \nu)} = \frac{1}{\Gamma(2k_2)} \sqrt{\frac{p \sinh \pi p}{2\pi^2}} \left[ \Gamma(k_1 + k_2 - \kappa) \Gamma(-k_1 + k_2 + \kappa) \times \Gamma(k_1 + k_2 + \kappa - 1) \Gamma(-k_1 + k_2 - \kappa + 1) \right]^{1/2} , \quad (2.28)$$

$k_1, k_2$  defined by:  $k_1 = \frac{1}{2}(1 \pm \nu)$ ,  $k_2 = \frac{1}{2}(1 \pm \eta)$ , where the correct sign depends on the boundary-conditions for  $r \rightarrow 0$  and  $r \rightarrow \infty$ , respectively. The number  $N_M$  denotes the maximal number of



states with  $0, 1, \dots, N_M < k_1 - k_2 - \frac{1}{2}$ .  $\kappa = k_1 - k_2 - n$  for the bound states and  $\kappa = \frac{1}{2}(1 + ip)$  for the scattering states.  ${}_2F_1(a, b; c; z)$  is the hypergeometric function [9, p.1057].

Note the zero-energy  $E_0 = 2\hbar^2/m$  which is a characteristic feature for the quantum motion on an hyperbolic space [20]. It has also been observed in [46] in terms of spherical coordinates, where the wave-functions and the spectrum were found by solving the Schrödinger equation.

### 3 The Equidistant Coordinate Systems

#### 3.1 Equidistant-I Coordinates

The first set of equidistant coordinates on HH(2) is given by

$$\left. \begin{aligned} z_1 &= \tanh \tau_1 e^{i\varphi_1} \\ z_2 &= \frac{\tanh \tau_1}{\cosh \tau_2} e^{i\varphi_2} \end{aligned} \right\} \quad (\tau_1, \tau_2 > 0, \varphi_1, \varphi_2 \in [0, 2\pi)) \quad . \quad (3.1)$$

This gives for the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \frac{1}{2m} \left[ p_{\tau_1}^2 + \frac{1}{\cosh^2 \tau_1} \left( p_{\tau_2}^2 + \frac{p_{\varphi_1}^2}{\sinh^2 \tau_2} - \frac{(p_{\varphi_1} + p_{\varphi_2})^2}{\cosh^2 \tau_2} \right) + \frac{p_{\varphi_2}^2}{\sinh^2 \tau_1} \right] \\ &= \frac{1}{2m} \left[ p_{\tau_1}^2 + \frac{p_{\tau_2}^2}{\cosh^2 \tau_1} + \frac{p_{\varphi_1}^2}{\cosh^2 \tau_1 \sinh^2 \tau_2 \cosh^2 \tau_2} \right. \\ &\quad \left. + \left( \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \right) p_{\varphi_2}^2 - \frac{2p_{\varphi_1} p_{\varphi_2}}{\cosh^2 \tau_1 \cosh^2 \tau_2} \right], \quad (3.2) \end{aligned}$$

and we obtain for the metric terms

$$(g^{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sinh^2 \tau_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\cosh^2 \tau_1 \sinh^2 \tau_2 \cosh^2 \tau_2} & -\frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \\ 0 & 0 & -\frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} & \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \end{pmatrix}, \quad (3.4)$$

$$\det(g_{ab}) = \sinh^2 \tau_1 \cosh^6 \tau_1 \sinh^2 \tau_2 \cosh^2 \tau_2. \quad (3.5)$$

Similarly as for the spherical system we introduce

$$(\hat{g}^{ab}) = \begin{pmatrix} \frac{1}{\cosh^2 \tau_1 \sinh^2 \tau_2 \cosh^2 \tau_2} & -\frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \\ -\frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} & \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1 \cosh^2 \tau_2} \end{pmatrix}, \quad (3.6)$$

and its inverse  $(\hat{g}_{ab})$

$$(\hat{g}_{ab}) = \sinh^2 \tau_1 \cosh^2 \tau_1 \sinh^2 \tau_2 \begin{pmatrix} \coth^2 \tau_1 \cosh^2 \tau_2 - 1 & 1 \\ 1 & \frac{1}{\sinh^2 \tau_2} \end{pmatrix}. \quad (3.7)$$

From these ingredients we find for the momentum operators

$$p_{\tau_1} = \frac{\hbar}{i} \left( \frac{\partial}{\partial \tau_1} + \frac{3}{2} \coth \tau_1 + \frac{1}{2} \tanh \tau_1 \right), \quad (3.8)$$

$$p_{\tau_2} = \frac{\hbar}{i} \left( \frac{\partial}{\partial \tau_2} + \frac{1}{2} (\coth \tau_2 + \tanh \tau_2) \right), \quad (3.9)$$

$$p_{\varphi_1} = \frac{\hbar}{i} \frac{\partial}{\partial \varphi_1}, \quad p_{\varphi_2} = \frac{\hbar}{i} \frac{\partial}{\partial \varphi_2}. \quad (3.10)$$

and the quantum potential according to our ordering prescription is found to read

$$\Delta V(\tau_1, \tau_2) = -\frac{\hbar^2}{8m} \left[ \left( \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1} - 16 \right) + \frac{1}{\cosh^2 \tau_1} \left( \frac{1}{\sinh^2 \tau_2} + \frac{1}{\cosh^2 \tau_2} \right) \right]. \quad (3.11)$$

From our line of reasoning of the spherical system, it is obvious that we can repeat the method to integrate out the ignorable coordinates  $(\varphi_1, \varphi_2)$  by means of Gaussian integrations. We find

$$\begin{aligned} & K(\tau_1'', \tau_1', \tau_2'', \tau_2', \varphi_1'', \varphi_1', \varphi_2'', \varphi_2'; T) \\ &= \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \int_{\varphi_1(t')=\varphi_1'}^{\varphi_1(t'')=\varphi_1''} \mathcal{D}\varphi_1(t) \int_{\varphi_2(t')=\varphi_2'}^{\varphi_2(t'')=\varphi_2''} \mathcal{D}\varphi_2(t) \sinh \tau_1 \cosh^3 \tau_1 \sinh \tau_2 \cosh \tau_2 \\ &\quad \times \exp \left( \frac{i}{\hbar} \int_0^T \left\{ \frac{m}{2} \left[ \dot{\tau}_1^2 + \cosh^2 \tau_1 \dot{\tau}_2^2 + (\dot{\varphi}_1, \dot{\varphi}_2)(\hat{g}_{ab}) \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} \right] \right. \right. \\ &\quad \left. \left. + \frac{\hbar^2}{8m} \left[ \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1} - 16 + \frac{1}{\cosh^2 \tau_1} \left( \frac{1}{\sinh^2 \tau_2} + \frac{1}{\cosh^2 \tau_2} \right) \right] \right\} dt \right) \\ &= \left( \frac{1}{16} \sinh 2\tau_1'' \sinh 2\tau_1' \sinh 2\tau_2'' \sinh 2\tau_2' \right)^{-1/2} e^{-2i\hbar T/m} \sum_{k_1, k_2 \in \mathbb{Z}^2} \frac{e^{ik_1(\varphi_1'' - \varphi_1') + ik_2(\varphi_2'' - \varphi_2')}}{(2\pi)^2} \\ &\quad \times K_{k_1 k_2}(\tau_1'', \tau_1', \tau_2'', \tau_2'; T) \end{aligned} \quad (3.12)$$

with the remaining path integral  $K_{k_1 k_2}(T)$  given by

$$\begin{aligned} K_{k_1 k_2}(\tau_1'', \tau_1', \tau_2'', \tau_2'; T) &= (\cosh \tau_1'' \cosh \tau_1')^{-1/2} \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \cosh \tau_1 \\ &\quad \times \exp \left( \frac{i}{\hbar} \int_0^T \left\{ \frac{m}{2} (\dot{\tau}_1^2 + \cosh^2 \tau_1 \dot{\tau}_2^2) \right. \right. \\ &\quad \left. \left. - \frac{\hbar^2}{2m} \left[ \frac{k_2^2 - \frac{1}{4}}{\sinh^2 \tau_1} + \frac{1}{\cosh^2 \tau_1} \left( \frac{k_1^2 - \frac{1}{4}}{\sinh^2 \tau_2} - \frac{(k_1 + k_2)^2 - \frac{1}{4}}{\cosh^2 \tau_2} + \frac{1}{4} \right) \right] \right\} dt \right). \end{aligned} \quad (3.13)$$

The path integration in  $\tau_1$  and  $\tau_2$  consists of two successive path integrations corresponding to two modified Pöschl–Teller potentials. In the  $\tau_2$ -path integration bound and continuous states are possible, which give rise to two expressions in the variable  $\tau_1$  (we set  $n_{\tau_2} = (|k_1 + k_2| - |k_1| - 2n - 1)$ ):

$$V_{k_2, n_{\tau_2}}(\tau_1) = \frac{\hbar^2}{2m} \left( \frac{k_2^2 - \frac{1}{4}}{\sinh^2 \tau_1} - \frac{n_{\tau_2}^2 - \frac{1}{4}}{\cosh^2 \tau_1} \right), \quad (3.14)$$

$$V_{k_2, k_{\tau_2}}(\tau_1) = \frac{\hbar^2}{2m} \left( \frac{k_2^2 - \frac{1}{4}}{\sinh^2 \tau_1} - \frac{-k_{\tau_2}^2 - \frac{1}{4}}{\cosh^2 \tau_1} \right). \quad (3.15)$$

Note that due to the potential trough in the variable  $\tau_2$  that there exist a number of bound states, labelled by  $n_{\tau_2}$  with  $n_{\tau_2} = 0, \dots, M_{\max}$ , where  $M_{\max} < [\frac{|k_2|-1}{2}]$  ( $[x]$  denotes the integer part of  $x$ ). Because the maximum number of states in the  $\tau_2$ -system is limited by  $|k_2|/2$ , there does not exist any bound states in the  $\tau_1$ -system. In the usual notation of the modified Pöschl–Teller functions we find the final solution in equidistant-I coordinates

$$\begin{aligned} & K(\tau_1'', \tau_1', \tau_2'', \tau_2', \varphi_1'', \varphi_1', \varphi_2'', \varphi_2'; T) \\ &= \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \int_0^\infty dk_{\tau_2} \int_0^\infty dp \Psi_{p, k_{\tau_2}, k_1, k_2}(\tau_1'', \tau_2'', \varphi_1'', \varphi_2'') \Psi_{p, k_{\tau_2}, k_1, k_2}^*(\tau_1', \tau_2', \varphi_1', \varphi_2') e^{-iE_p T/\hbar} \\ &+ \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \sum_{n_{\tau_2}=0, \dots, M_{\max}} \int_0^\infty dp \Psi_{p, n_{\tau_2}, k_1, k_2}(\tau_1'', \tau_2'', \varphi_1'', \varphi_2'') \Psi_{p, n_{\tau_2}, k_1, k_2}^*(\tau_1', \tau_2', \varphi_1', \varphi_2') e^{-iE_p T/\hbar}, \end{aligned} \quad (3.16)$$

with the wave-functions and the energy-spectrum given by

$$\Psi_{p, k_{\tau_2}, k_1, k_2}(\tau_1, \tau_2, \varphi_1, \varphi_2) = \left( \frac{1}{4} \sinh 2\tau_1 \sinh 2\tau_2 \right)^{-1/2} \frac{e^{i(k_1 \varphi_1 + k_2 \varphi_2)}}{2\pi} \Psi_{k_{\tau_2}}^{(k_1, k_1 + k_2)}(\beta) \Psi_p^{(k_1, i k_{\tau_2})}(\tau_1), \quad (3.17)$$

$$\Psi_{p, n_{\tau_2}, k_1, k_2}(\tau_1, \tau_2, \varphi_1, \varphi_2) = \left( \frac{1}{4} \sinh 2\tau_1 \sinh 2\tau_2 \right)^{-1/2} \frac{e^{i(k_1 \varphi_1 + k_2 \varphi_2)}}{2\pi} \Psi_{n_{\tau_2}}^{(k_1, k_1 + k_2)}(\beta) \Psi_p^{(k_1, n_{\tau_2})}(\tau_1), \quad (3.18)$$

$$E_p = \frac{\hbar^2}{2m} (p^2 + 4). \quad (3.19)$$

The spectrum is the same as in spherical system, as it should be.

### 3.2 Equidistant-II Coordinates

The second set of equidistant coordinates is given by

$$\left. \begin{aligned} z_1 &= \frac{i \sinh \tau_2 \cosh u - \cosh \tau_2 \sinh u}{i \cosh \tau_2 \cosh u + \sinh \tau_2 \sinh u} \\ z_2 &= \frac{i \tanh \tau_1}{i \cosh \tau_2 \cosh u + \sinh \tau_2 \sinh u} e^{i\varphi} \end{aligned} \right\} \quad (\tau_1 > 0, \tau_2 \in \mathbb{R}, u \in \mathbb{R}, \varphi \in [0, 2\pi)) \quad (3.20)$$

This gives for the Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \left[ p_{\tau_1}^2 + \frac{1}{\cosh^2 \tau_1} \left( p_{\tau_2}^2 + \frac{p_u^2 - p_\varphi^2}{\cosh^2 2\tau_2} - \frac{2 \sinh 2\tau_2}{\cosh^2 2\tau_2} p_u p_\varphi \right) + \frac{p_\varphi^2}{\sinh^2 \tau_1} \right] \quad (3.21)$$

$$= \frac{1}{2m} \left[ p_{\tau_1}^2 + \frac{p_{\tau_2}^2}{\cosh^2 \tau_1} + \frac{p_u^2}{\cosh^2 \tau_1 \cosh^2 2\tau_2} + \left( \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1 \cosh^2 2\tau_2} \right) p_\varphi^2 - \frac{2 \sinh 2\tau_2}{\cosh^2 2\tau_2} p_u p_\varphi \right], \quad (3.22)$$

and we obtain for the metric terms

$$(g^{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\cosh^2 \tau_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\cosh^2 \tau_1 \cosh^2 2\tau_2} & -\frac{\sinh 2\tau_2}{\cosh^2 \tau_1 \cosh^2 2\tau_2} \\ 0 & 0 & -\frac{\sinh 2\tau_2}{\cosh^2 \tau_1 \cosh^2 2\tau_2} & \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1 \cosh^2 2\tau_2} \end{pmatrix}, \quad (3.23)$$

$$\det(g_{ab}) = \sinh^2 \tau_1 \cosh^6 \tau_1 \cosh^2 2\tau_2. \quad (3.24)$$

Similarly as for the equidistant-I system, the quantum potential according to our ordering prescription is found to read

$$\Delta V(\tau_1) = -\frac{\hbar^2}{8m} \left( \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1} - 16 \right), \quad (3.25)$$

and it does not depend on  $\tau_2$ . We can therefore write down the path integral, again we separate off the ignorable coordinates  $(u, \varphi)$ , and we obtain

$$\begin{aligned} & K(\tau_1'', \tau_1', \tau_2'', \tau_2', u'', u', \varphi'', \varphi'; T) \\ &= \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \int_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \sinh \tau_1 \cosh^3 \tau_1 \cosh 2\tau_2 \\ & \quad \times \exp \left[ \frac{i}{\hbar} \int_0^T \left( \frac{m}{2} \left\{ \dot{\tau}_1^2 + \cosh^2 \tau_1 \left[ \dot{\tau}_2^2 + (\dot{u}, \dot{\varphi})(\hat{g}_{ab}) \left( \frac{\dot{u}}{\dot{\varphi}} \right) \right] \right\} - \Delta V(\tau_1) \right) dt \right] \\ &= \left( \frac{1}{4} \sinh 2\tau_1'' \sinh 2\tau_1' \cosh 2\tau_2'' \cosh 2\tau_2' \right)^{-1/2} e^{-2i\hbar T/m} \sum_{k_\varphi \in \mathbb{Z}} \int dk_u \frac{e^{ik_u(u''-u') + ik_\varphi(\varphi''-\varphi')}}{(2\pi)^2} \\ & \quad \times K_{k_u k_\varphi}(\tau_1'', \tau_1', \tau_2'', \tau_2'; T) \end{aligned} \quad (3.26)$$

with the path integral  $K_{k_u k_\varphi}(T)$  given by

$$\begin{aligned} K_{k_u k_\varphi}(\tau_1'', \tau_1', \tau_2'', \tau_2'; T) &= (\cosh \tau_1'' \cosh \tau_1')^{-1/2} \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \cosh \tau_1 \\ & \quad \times \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} (\dot{\tau}_1^2 + \cosh^2 \tau_1 \dot{\tau}_2^2) \right. \right. \\ & \quad \left. \left. - \frac{\hbar^2}{2m} \frac{k_\varphi^2 - \frac{1}{4}}{\sinh^2 \tau_1} - \frac{\hbar^2}{2m \cosh^2 \tau_1} \left( \frac{k_u^2 - k_\varphi^2}{\cosh^2 2\tau_2} - \frac{2 \sinh 2\tau_2}{\cosh^2 2\tau_2} k_u k_\varphi + \frac{1}{4} \right) \right] dt \right\}. \end{aligned} \quad (3.27)$$

The potential

$$V^{(\text{HBP})}(\tau_2) = \frac{\hbar^2}{2m \cosh^2 \tau_1} \left( \frac{k_u^2 - k_\varphi^2}{\cosh^2 2\tau_2} - 2k_u k_\varphi \frac{\tanh 2\tau_2}{\cosh 2\tau_2} \right)$$

$$\begin{aligned}
&= \frac{\hbar^2}{2m \cosh^2 \tau_1} \left( (k_u^2 - k_\varphi^2) - (k_u^2 - k_\varphi^2) \tanh^2 2\tau_2 - 2k_u k_\varphi \frac{\tanh 2\tau_2}{\cosh 2\tau_2} \right) \\
&\equiv \frac{\hbar^2}{2m} \left( V_0 + V_1 \frac{\tanh 2\tau_2}{\cosh 2\tau_2} + V_2 \tanh^2 2\tau_2 \right)
\end{aligned} \tag{3.28}$$

is called hyperbolic barrier potential [40]. The corresponding path integral can be found in Ref. [12, 21] (and references therein) by means of the coordinate transformation

$$\frac{1 + i \sinh 2\tau_2}{2} = \cosh^2 z . \tag{3.29}$$

We set  $1 + \lambda \equiv \sqrt{V_2 - iV_1 + \frac{1}{4}}$ ,  $\lambda_{R,I} = (\Re, \Im)(\lambda)$ ,  $n = 0, 1, \dots, N_M < [\lambda_R - \frac{1}{2}]$ . The discrete wave-functions have the form

$$\begin{aligned}
\Psi_n^{(\text{HBP})}(\tau_2) &= \left[ \frac{(2\lambda_R - 2n - 1)n! \Gamma(\lambda - n)}{2\Gamma(2\lambda_R - n)\Gamma(n + 1 - \lambda^*)} \right]^{1/2} \\
&\quad \times \left( \frac{1 + i \sinh x}{2} \right)^{\frac{1}{2}(\frac{1}{2} - \lambda)} \left( \frac{1 - i \sinh x}{2} \right)^{\frac{1}{2}(\frac{1}{2} - \lambda^*)} P_n^{(-\lambda^*, -\lambda)}(i \sinh 2\tau_2) ,
\end{aligned} \tag{3.30}$$

$$E_n = -\frac{\hbar^2}{2m} n_{\tau_2}^2 \tag{3.31}$$

$$n_{\tau_2} = n + \frac{1}{2} - \sqrt{\frac{1}{2} \left[ \sqrt{(\frac{1}{4} + V_2)^2 + V_1^2} + \frac{1}{4} + V_2 \right]} , \tag{3.32}$$

The continuous wave-functions are

$$\begin{aligned}
\Psi_{k_{\tau_2}}^{(\text{HBP})}(\tau_2) &= \frac{\Gamma(\frac{1}{2} - \lambda_R - i k_{\tau_2})}{\pi \Gamma(1 - \lambda^*)} \sqrt{k_{\tau_2} \sinh(2\pi k_{\tau_2}) \Gamma\left(\frac{1}{2} + i(k_{\tau_2} - \lambda_I)\right) \Gamma\left(\frac{1}{2} + i(k_{\tau_2} + \lambda_I)\right)} \\
&\quad \times {}_2F_1\left(\frac{1}{2} + i(\lambda_I - k_{\tau_2}), \frac{1}{2} - \lambda_R - i k_{\tau_2}; 1 - \lambda^*; \frac{i \sinh 2\tau_2 - 1}{i \sinh 2\tau_2 + 1}\right) ,
\end{aligned} \tag{3.33}$$

with  $E_{k_{\tau_2}} = \frac{\hbar^2 k_{\tau_2}^2}{2m}$ . The emerging path integral in the variable  $\tau_1$  is of almost the same form as in the case of equidistant-I coordinates, only continuous states are allowed, c.f. the discussion after (3.15), and we find for the final solution:

$$\begin{aligned}
&K(\tau_1'', \tau_1', \tau_2'', \tau_2', u'', u', \varphi'', \varphi'; T) \\
&= \sum_{k_\varphi \in \mathbb{Z}} \int dk_u \int_0^\infty dk_{\tau_2} \int_0^\infty dp \Psi_{p, k_{\tau_2}, k_u, k_\varphi}(\tau_1'', \tau_2'', u'', \varphi'') \Psi_{p, k_{\tau_2}, k_u, k_\varphi}^*(\tau_1', \tau_2', u', \varphi') e^{-iE_p T/\hbar} \\
&+ \sum_{k_\varphi \in \mathbb{Z}} \int dk_u \sum_{n_{\tau_2}=0}^{M_{\max}} \int_0^\infty dp \Psi_{p, n_{\tau_2}, k_u, k_\varphi}(\tau_1'', \tau_2'', u'', \varphi'') \Psi_{p, n_{\tau_2}, k_u, k_\varphi}^*(\tau_1', \tau_2', u', \varphi') e^{-iE_p T/\hbar} ,
\end{aligned} \tag{3.34}$$

with the wave-functions and the energy-spectrum given by

$$\Psi_{p, k_{\tau_2}, k_u, k_\varphi}(\tau_1, \tau_2, u, \varphi) = \left(\frac{1}{2} \sinh 2\tau_1 \cosh 2\tau_2\right)^{-1/2} \frac{e^{i(k_u u + k_\varphi \varphi)}}{2\pi} \Psi_{k_{\tau_2}}^{(\text{HBP})}(\tau_2) \Psi_p^{(k_u, i k_{\tau_2})}(\tau_1) , \tag{3.35}$$

$$\Psi_{p,n\tau_2,k_u,k_\varphi}(\tau_1,\tau_2,u,\varphi) = \left(\frac{1}{2}\sinh 2\tau_1 \cosh 2\tau_2\right)^{-1/2} \frac{e^{i(k_u u + k_\varphi \varphi)}}{2\pi} \Psi_{n\tau_2}^{(\text{HBP})}(\tau_2) \Psi_p^{(k_u,n\tau_2)}(\tau_1) , \quad (3.36)$$

$$E_p = \frac{\hbar^2}{2m}(p^2 + 4) . \quad (3.37)$$

The spectrum is the same as before, as it should be.

### 3.3 Equidistant-III Coordinates

The third set of equidistant coordinates is given by

$$\left. \begin{aligned} z_1 &= \frac{\sinh \tau_2 + iu e^{-\tau_2}}{\cosh \tau_2 + iu e^{-\tau_2}} \\ z_2 &= \frac{\tanh \tau_1}{\cosh \tau_2 + iu e^{-\tau_2}} e^{i\varphi} \end{aligned} \right\} \quad (\tau_1, \tau_2 > 0, u \in \mathbb{R}, \varphi \in [0, 2\pi)) . \quad (3.38)$$

This gives for the Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \left[ p_{\tau_1}^2 + \frac{1}{\cosh^2 \tau_1} (p_{\tau_2}^2 + (e^{2\tau_2} p_u + p_\varphi)^2 - p_\varphi^2) + \frac{p_\varphi^2}{\sinh^2 \tau_1} \right] \quad (3.39)$$

$$= \frac{1}{2m} \left[ p_{\tau_1}^2 + \frac{p_{\tau_2}^2}{\cosh^2 \tau_1} + \frac{e^{4\tau_2}}{\cosh^2 \tau_1} p_u^2 + \frac{p_\varphi^2}{\sinh^2 \tau_1} + \frac{2e^{2\tau_2}}{\cosh^2 \tau_1} p_\varphi p_u \right] , \quad (3.40)$$

and we obtain for the metric terms

$$(g^{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\cosh^2 \tau_1} & 0 & 0 \\ 0 & 0 & \frac{e^{4\tau_2}}{\cosh^2 \tau_1} & \frac{e^{2\tau_2}}{\cosh^2 \tau_1} \\ 0 & 0 & \frac{e^{2\tau_2}}{\cosh^2 \tau_1} & \frac{1}{\sinh^2 \tau_1} \end{pmatrix} , \quad (3.41)$$

$$\det(g_{ab}) = e^{-4\rho} \sinh^2 \tau_1 \cosh^6 \tau_1 . \quad (3.42)$$

The quantum potential according to our ordering prescription is found to read

$$\Delta V(\tau_1) = \frac{2\hbar^2}{m} - \frac{\hbar^2}{8m} \left( \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1} \right) . \quad (3.43)$$

We can therefore write down the path integral, again we separate off the ignorable coordinates  $(u, \varphi)$ , and we obtain

$$\begin{aligned} & K(\tau_1'', \tau_1', \tau_2'', \tau_2', u'', u', \varphi'', \varphi'; T) \\ &= \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \int_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) \sinh \tau_1 \cosh^3 \tau_1 \end{aligned}$$

$$\begin{aligned}
& \times \exp \left[ \frac{i}{\hbar} \int_0^T \left( \frac{m}{2} \left\{ \dot{\tau}_1^2 + \cosh^2 \tau_1 \left[ \dot{\tau}_2^2 + e^{-4\tau_2} \cosh^2 \tau_1 \dot{u}^2 + \sinh^2 \dot{\varphi}^2 - 2e^{-4\tau_2} \sinh^2 \tau_1 \dot{u} \dot{\varphi} \right] \right\} \right. \right. \\
& \quad \left. \left. + \frac{\hbar^2}{8m} \left( \frac{1}{\sinh^2 \tau_1} - \frac{1}{\cosh^2 \tau_1} - 16 \right) \right) dt \right] \\
& = \left( \frac{1}{4} \sinh 2\tau_1'' \sinh 2\tau_1' \right)^{-1/2} e^{\tau_2' + \tau_2''} e^{-2i\hbar T/m} \\
& \quad \times \sum_{k_\varphi \in \mathbb{Z}} \frac{e^{ik_\varphi(\varphi'' - \varphi')}}{2\pi} \int dk_u \frac{e^{ik_u(u'' - u')}}{2\pi} K_{k_u k_\varphi}(\tau_1'', \tau_1', \tau_2'', \tau_2'; T)
\end{aligned} \tag{3.44}$$

with the path integral  $K_{k_u k_\varphi}(T)$  given by

$$\begin{aligned}
K_{k_u k_\varphi}(\tau_1'', \tau_1', \tau_2'', \tau_2'; T) & = (\cosh \tau_1'' \cosh \tau_1')^{-1/2} \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \int_{\tau_2(t')=\tau_2'}^{\tau_2(t'')=\tau_2''} \mathcal{D}\tau_2(t) \cosh \tau_1 \\
& \times \exp \left( \frac{i}{\hbar} \int_0^T \left\{ \frac{m}{2} (\dot{\tau}_1^2 + \cosh^2 \tau_1 \dot{\tau}_2^2) - \frac{\hbar^2}{2m} \left[ \frac{k_u^2}{\cosh^2 \tau_1} \left( e^{4\rho} + 2 \frac{k_\varphi}{|k_u|} e^{2\rho} + \frac{1}{4} \right) + \frac{k_\varphi^2 - \frac{1}{4}}{\sinh^2 \tau_1} \right] \right\} dt \right).
\end{aligned} \tag{3.45}$$

This is a path integral which is related to the Morse potential, respectively to the oscillator-like potential on the hyperbolic plane [10, 21] (and references therein), where with respect to the variable  $\tau_2$  discrete and continuous states are allowed. We have

$$\begin{aligned}
K_{k_u k_\varphi}(\tau_1'', \tau_1', \tau_2'', \tau_2'; T) & = \int dk_{\tau_2} \Psi_{p_{\tau_2}}^{(\text{MP})}(\tau_2'') \Psi_{p_{\tau_2}}^{(\text{MP})*}(\tau_2') \\
& \times \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \dot{\tau}_1^2 - \frac{\hbar^2}{2m} \left( \frac{k_\varphi^2 - \frac{1}{4}}{\sinh^2 \tau_1} - \frac{-k_{\tau_2}^2 - \frac{1}{4}}{\cosh^2 \tau_1} \right) \right] dt \right\} \\
& + \sum_{n_{\tau_2}} \Psi_{n_{\tau_2}}^{(\text{MP})}(\tau_2'') \Psi_{n_{\tau_2}}^{(\text{MP})}(\tau_2') \\
& \times \int_{\tau_1(t')=\tau_1'}^{\tau_1(t'')=\tau_1''} \mathcal{D}\tau_1(t) \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \dot{\tau}_1^2 - \frac{\hbar^2}{2m} \left( \frac{k_\varphi^2 - \frac{1}{4}}{\sinh^2 \tau_1} + \frac{n_{\tau_2}^2 - \frac{1}{4}}{\cosh^2 \tau_1} \right) \right] dt \right\}.
\end{aligned} \tag{3.46}$$

The Morse potential wave-functions have the form  $(k_{\varphi, k_u} = k_\varphi \text{sign}(k_u), n_{\tau_2} = k_{\varphi, k_u} - 2n - 1)$ ,  $n < [\frac{1}{2}(k_{\varphi, k_u} - 2n - 1)]$

$$\Psi_{n_{\tau_2}}^{(\text{MP})}(\tau_2) = \sqrt{\frac{2n!(k_{\varphi, k_u} - 2n_{\tau_2} - 1)}{\Gamma(k_{\varphi, k_u} - n)}} (|k_u| e^{2\tau_2})^{k_{\varphi, k_u} - n} e^{-\frac{1}{2}|k_u| e^{2\tau_2}} L_n^{(k_{\varphi, k_u} - 2n - 1)}(|k_u| e^{2\tau_2}). \tag{3.47}$$

$$\Psi_{k_{\tau_2}}^{(\text{MP})}(\tau_2) = \sqrt{\frac{k_{\tau_2} \sinh \pi k_{\tau_2}}{2\pi^2 |k_u|}} \Gamma[\frac{1}{2}(1 + ik_{\tau_2} + k_{\varphi, k_u})] W_{k_{\varphi, k_u}/2, k_{\tau_2}/2}(|k_u| e^{2\tau_2}). \tag{3.48}$$

The bound states can only exists for  $k_{\varphi, k_u} > 0$ . Since  $\max(n_{\tau_2}) \leq |k_\varphi|/2$  (c.f. the discussion following (3.15)), only continuous states are allowed with respect to the variable  $\tau_1$ . Therefore

we get finally, where  $E_p$  is the same as in the previous equidistant systems:

$$\begin{aligned}
K(\tau_1'', \tau_1', \tau_2'', \tau_2', u'', u', \varphi'', \varphi'; T) &= (\sinh \tau_1' \cosh \tau_1' \sinh \tau_1'' \cosh \tau_1'')^{-1/2} e^{(\tau_2' + \tau_2'')/2} \\
&\times \sum_{k_\varphi \in \mathbb{Z}} \int dk_u \frac{e^{i[k_\varphi(\varphi'' - \varphi') + k_u(u'' - u')]} (2\pi)^2}{\int_0^\infty dp e^{-iE_p T/\hbar}} \\
&\times \left\{ \sum_{n_{\tau_2}} \int_0^\infty dp \Psi_{n_{\tau_2}}(\tau_2'')^{(\text{MP})} \Psi_{n_{\tau_2}}^{(\text{MP})}(\tau_2') \Psi_p^{(n_{\tau_2}, k_\varphi)*}(\tau_1'') \Psi_p^{(n_{\tau_2}, k_\varphi)}(\tau_1') \right. \\
&\quad \left. + \int_0^\infty dk_{\tau_2} \int_0^\infty dp \Psi_{k_{\tau_2}}^{(\text{MP})*}(\tau_2'') \Psi_{k_{\tau_2}}^{(\text{MP})}(\tau_2') \Psi_p^{(ik_{\tau_2}, k_\varphi)*}(\tau_1'') \Psi_p^{(ik_{\tau_2}, k_\varphi)}(\tau_1') \right\}. \quad (3.49)
\end{aligned}$$

This concludes the discussion of the three equidistant systems.

## 4 The Horicyclic Coordinate Systems

### 4.1 Horicyclic-I Coordinates

The first set of horicyclic coordinates is given by

$$\left. \begin{aligned} z_1 &= \frac{-1 + e^{2q} + r^2 + 2iu}{1 + e^{2q} + r^2 + 2iu} \\ z_2 &= \frac{r}{1 + e^{2q} + r^2 + 2iu} e^{i\varphi} \end{aligned} \right\} \quad (q \in \mathbb{R}, r > 0, u \in \mathbb{R}, \varphi \in [0, 2\pi)) . \quad (4.1)$$

This gives for the Hamiltonian

$$\mathcal{H} = \frac{1}{2m} \left\{ p_q^2 + e^{2q} \left[ p_r^2 + \left( \frac{p_\varphi}{r} + r p_u \right)^2 \right] + e^{4q} p_u^2 \right\} \quad (4.2)$$

$$= \frac{1}{2m} \left[ p_q^2 + e^{2q} p_r^2 + e^{2q} (e^{2q} + r^2) p_u^2 + \frac{e^{2q}}{r^2} p_\varphi^2 + 2e^{2q} p_\varphi p_u \right], \quad (4.3)$$

and we obtain for the metric terms

$$(g^{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2q} & 0 & 0 \\ 0 & 0 & e^{2q}(e^{2q} + r^2) & e^{2q} \\ 0 & 0 & e^{2q} & e^{2q}/r^2 \end{pmatrix}, \quad (4.4)$$

$$\det(g_{ab}) = r^2 e^{-8q}. \quad (4.5)$$

Therefore we obtain for the path integral and Gaussian path integration in  $(u, \varphi)$

$$\begin{aligned}
&K(q'', q', r'', r', u'', u', \varphi'', \varphi'; T) \\
&= e^{-2i\hbar T/m} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(t) \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \int_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int_{\varphi(t')=\varphi'}^{\varphi(t'')=\varphi''} \mathcal{D}\varphi(t) r e^{-4q}
\end{aligned}$$



$$\begin{aligned}
& \times \exp \left( \frac{i}{\hbar} \int_0^T \left\{ \frac{m}{2} \left[ \dot{q}^2 + \frac{\dot{r}^2}{e^{2q}} - e^{-4q} (\dot{u}^2 + (e^{2q} + r^2) \dot{\varphi}^2 - 2r^2 \dot{u} \dot{\varphi}) \right] + e^{2q} \frac{\hbar^2}{2mr^2} \right\} dt \right) \\
& = e^{q'+q''} e^{-2i\hbar T/m} \int_0^\infty dk_u \frac{e^{ik_u(u''-u')}}{2\pi} \sum_{k_\varphi \in \mathbb{Z}} \frac{e^{ik_\varphi(\varphi''-\varphi')}}{2\pi} K_{k_u k_\varphi}(q'', q', r'', r'; T) , \quad (4.6)
\end{aligned}$$

with the remaining path integral  $K_{k_u k_\varphi}(T)$  given by

$$\begin{aligned}
K_{k_u k_\varphi}(q'', q', r'', r'; T) &= (r' r'')^{-1/2} e^{(q'+q'')/2} e^{i\hbar T/8m} \int_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y^2} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \\
&\times \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \left( \frac{\dot{r}^2 + \dot{y}^2}{y^2} \right) - y^2 \frac{\hbar^2}{2m} \left( \frac{k_\varphi^2 - \frac{1}{4}}{r^2} + r^2 k_u^2 + 2k_u k_\varphi \right) - y^4 \frac{\hbar^2 k_u^2}{2m} \right] dt \right\} \quad (4.7) \\
&= |k_u| e^{i\hbar T/8m} \sum_{n=0}^\infty \frac{2n!}{\Gamma(n + |k_u| + 1)} (|k_u| r' r'')^{|k_u|} e^{-\frac{1}{2}|k_u|(r'^2 + r''^2)} L_n^{(|k_u|)}(|k_u| r') L_n^{(|k_u|)}(|k_u| r'') \\
&\times \int_{y(t')=y'}^{y(t'')=y''} \frac{\mathcal{D}y(t)}{y} \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \frac{\dot{y}^2}{y^2} - \frac{\hbar^2 y^2}{2m} (|k_u|(4n + 2|k_\varphi| + 2) + 2k_u k_\varphi + k_u^2 y^2) \right] dt \right\} . \quad (4.8)
\end{aligned}$$

I have used the path integral solution of the radial harmonic oscillator [21, 41] ( $L_n^{(k_u)}(z)$  denote Laguerre polynomials), and we have also performed the transformation  $q = \ln y$  according to [21]. In particular, the path integral (4.7) is of the form called “oscillator-like” potential on the hyperbolic plane. The “oscillator-like” term reads  $k_u^2 y^2$ . The potential in  $r$  has the form

$$\frac{\hbar^2}{2m} \left( \frac{k_\varphi^2 - \frac{1}{4}}{r^2} + r^2 k_u^2 + 2k_u k_\varphi \right).$$

Due to the spectrum of the radial harmonic oscillator we see that  $|k_u|(4n + 2|k_\varphi| + 2) + 2k_u k_\varphi \geq 0$  and therefore only continuous states are allowed in (4.7) which is, of course, related to the Morse potential. Using the result of [21] we finally obtain the solution of the horicyclic-I coordinates on  $\text{HH}(2)$  as follows

$$\begin{aligned}
& K(q'', q', r'', r', u'', u', \varphi'', \varphi'; T) \\
&= \int_0^\infty dk_u \sum_{k_\varphi \in \mathbb{Z}} \sum_{n=0}^\infty \int_0^\infty dp e^{iE_p T/\hbar} \Psi_{p,n,k_u,k_\varphi}(q'', r'', u'', \varphi'') \Psi_{p,n,k_u,k_\varphi}^*(q', r', u', \varphi') , \quad (4.9)
\end{aligned}$$

$$E_p = \frac{\hbar^2}{2m} (p^2 + 4) . \quad (4.10)$$

with the wave-functions given by (we abbreviate  $E_n = 2n + |k_\varphi| + \text{sign}(k_u)k_\varphi + 1$ )

$$\begin{aligned}
\Psi_{p,n,k_u,k_\varphi}(y, r, u, \varphi) &= \frac{e^{i(k_u u + k_\varphi \varphi)}}{2\pi} \sqrt{\frac{2n!}{\Gamma(n + |k_u| + 1)}} (|k_u| r)^{|k_u|} e^{-\frac{1}{2}|k_u| r^2} L_n^{(|k_u|)}(|k_u| r) \\
&\times \sqrt{\frac{p \sinh \pi p}{2\pi^2}} \Gamma(\tfrac{1}{2}(1 + ip + E_n)) W_{-E_n/2, ip/2}(|k_u| e^{2q}) . \quad (4.11)
\end{aligned}$$

## 4.2 Horicyclic-II Coordinates

The second set of horicyclic coordinates is given by

$$\left. \begin{aligned} z_1 &= \frac{2(u+xz) - i(e^{2q} + x^2 + z^2 - 1)}{2(u+xz) - i(e^{2q} + x^2 + z^2 + 1)} \\ z_2 &= \frac{-2(z+ix)}{2(u+xz) - i(e^{2q} + x^2 + z^2 + 1)} \end{aligned} \right\} \quad (q, u, x, z \in \mathbb{R}) . \quad (4.12)$$

This gives in the usual way for the Hamiltonian and for the metric terms

$$\mathcal{H} = \frac{1}{2m} \left\{ p_q^2 + e^{2q} \left[ p_x^2 + (p_z - 2xp_u)^2 \right] + e^{4q} p_u^2 \right\} \quad (4.13)$$

$$= \frac{1}{2m} \left[ p_q^2 + e^{2q} \left( p_x^2 + p_z^2 + (4x^2 + e^{2q}) p_u^2 - 4xp_z p_u \right) \right] \quad (4.14)$$

$$(g^{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2q} & 0 & 0 \\ 0 & 0 & e^{2q} & -2xe^{2q} \\ 0 & 0 & -2xe^{2q} & 4x^2 e^{2q} + e^{4q} \end{pmatrix} , \quad (4.15)$$

$$\det(g_{ab}) = e^{-8q} . \quad (4.16)$$

We can write down the path integral and separate off the  $u$  and  $z$  path integration by means of Gaussian integrations yielding

$$\begin{aligned} & K(q'', q', x'', x', u'', u', z'', z'; T) \\ &= e^{-2i\hbar T/m} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(t) \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int_{u(t')=u'}^{u(t'')=u''} \mathcal{D}u(t) \int_{z(t')=z'}^{z(t'')=z''} \mathcal{D}z(t) e^{-4q} \\ & \quad \times \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \left( \dot{q}^2 + \frac{\dot{x}^2}{e^{2q}} \right) + e^{-4q} \left( (4x^2 + e^{2q}) \dot{z}^2 + \dot{u}^2 + 4x\dot{u}\dot{z} \right) \right] dt \right\} \\ &= e^{-2i\hbar T/m} e^{q'+q''} \int dk_u \int dk_z \frac{e^{ik_u(u''-u') + ik_z(z''-z')}}{(4\pi)^2} K_{k_u k_z}(q'', q', x'', x'; T) \end{aligned} \quad (4.17)$$

with the remaining path integral  $K_{k_u k_z}(T)$

$$\begin{aligned} & K_{k_u k_z}(q'', q', x'', x'; T) = e^{-(q'+q'')/2} \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(t) \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) e^{-q} \\ & \quad \times \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \left( \dot{q}^2 + \frac{\dot{x}^2}{e^{2q}} \right) - \frac{\hbar^2 k_u^2}{2m} e^{2q} \left( e^{2q} + 4 \left( x - \frac{k_z}{2k_u} \right)^2 \right) \right] dt \right\} \\ &= \sum_{n=0}^{\infty} \Psi_n^{(\text{HO})}(x') \Psi_n^{(\text{HO})}(x'') \\ & \quad \times \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(t) \exp \left\{ \frac{i}{\hbar} \int_0^T \left[ \frac{m}{2} \left[ \dot{q}^2 - \frac{\hbar^2}{2m} e^{2q} \left( k_u^2 e^{2q} + 4|k_u|(n + \frac{1}{2}) \right) \right] \right] dt \right\} . \end{aligned} \quad (4.18)$$

The  $\Psi_n(x)$  are the wave-functions of the harmonic oscillator with frequency  $\omega = 2\hbar|k_u|/m$  shifted by  $-k_z/2k_u$  and are given by

$$\Psi_n^{(\text{HO})}(x) = \frac{\sqrt[4]{2|k_u|/\pi}}{\sqrt{2^n n!}} e^{-|k_u|x^2} H_n \left[ 2|k_u| \left( x - \frac{k_z}{2|k_u|} \right) \right]. \quad (4.19)$$

Therefore we obtain for the complete solution in horicyclic-II coordinates

$$\begin{aligned} & K(q'', q', x'', x', u'', u', z'', z'; T) \\ &= \int_0^\infty dk_u \int_0^\infty dk_u \sum_{n=0}^\infty \int_0^\infty dp e^{-iE_p T/\hbar} \Psi_{p,n,k_u,k_\varphi}(q'', x'', u'', z'') \Psi_{p,n,k_u,k_\varphi}^*(q', x', u', z'), \end{aligned} \quad (4.20)$$

$$\begin{aligned} \Psi_{q,n,k_u,k_z}(y, x, u, z) &= \frac{e^{i(k_u u + k_z z)}}{2\pi} \Psi_n^{(\text{HO})}(x) \\ &\times \sqrt{\frac{p \sinh \pi p}{4\pi^2 |k_u|}} \Gamma\left[\frac{1}{2}(1 + ip + n + \frac{1}{2})\right] W_{-(n+\frac{1}{2})/2, ip/2}(2|k_u|e^{2q}), \end{aligned} \quad (4.21)$$

$$E_p = \frac{\hbar^2}{2m}(p^2 + 4). \quad (4.22)$$

This concludes the discussion.

## 5 The Remaining Coordinate Systems

In this section we enumerate the remaining six coordinates on HH(2) for completeness. Three of them are parametric coordinate systems. i.e. an additional parameter, say  $a$ , is given, which for instance describes the interfocal distance of an ellipse. We do not formulate the path integral, because these coordinate system are quite involved. The knowledge of the corresponding special functions which are solutions of the Helmholtz, respectively the Schrödinger equation, is very limited.

There are also parabolic coordinates, where we formulate the path integral for the semicircular parabolic system, however, as for the parametric systems, the corresponding path integral cannot be solved.

### 5.1 The Parametric Coordinate Systems

We briefly sketch the three parametric coordinate systems. They are

1. [Elliptic-I Coordinates.] We have the following representation

$$z_1^2 = \frac{a(\nu-1)(\varrho-1)}{(a-1)\nu\varrho} e^{2i\varphi_1}, \quad z_2^2 = \frac{(\nu-a)(a-\varrho)}{(a-1)\nu\varrho} e^{2i\varphi_2}, \quad (5.1)$$

with  $1 \leq \varrho \leq a \leq \nu < \infty$ ,  $a > 1$ .

2. [Elliptic-II Coordinates.] Elliptic-II coordinates are given by

$$z_1^2 = -\frac{(a-1)\nu\varrho}{a(\nu-1)(1-\varrho)} e^{2i\varphi_1}, \quad z_2^2 = \frac{(\nu-a)(a-\varrho)}{a(\nu-1)(1-\varrho)} e^{2i\varphi_2}, \quad (5.2)$$

with  $1 < a \leq \nu$ ,  $\varrho \leq 0$ , and  $0 < a-1 \leq 1$ .

3. [Semi-hyperbolic Coordinates.] Semi-hyperbolic coordinates are given by

$$\left. \begin{aligned} z_1 &= \frac{is_1 \cosh u - s_0 \sinh u}{is_0 \cosh u + s_1 \sinh u} \\ z_2 &= \frac{is_2 e^{i\varphi}}{is_0 \cosh u + s_1 \sinh u} \end{aligned} \right\} \quad (u \in \mathbb{R}, \varphi \in [0, 2\pi)) . \quad (5.3)$$

$s_0, s_1, s_2$  are defined by

$$\frac{1}{2}(s_0 + is_1)^2 = \frac{(\nu - a)(\varrho - a)}{a(a - a^*)}, \quad s_2^2 = -\frac{\nu\varrho}{|a|^2}, \quad (5.4)$$

with  $\nu < 0 < \varrho$ ,  $a = \alpha + i\beta$  ( $\alpha, \beta \in \mathbb{R}$ ).

We do not go into the details of these coordinate systems. Let us only mention that for the corresponding coordinate systems on the two-dimensional hyperboloid, from where the systems on HH(2) have their notion, the corresponding solutions of the Schrödinger equation are known as Lamé-Wagnerian functions, see [15, 28] for details.

## 5.2 The Elliptic- and Hyperbolic-Parabolic Coordinate Systems

For the last three coordinate systems we use a notation which differs from [2] and is more in accordance with our publications [15, 18, 19]. First we define the elliptic-parabolic coordinate system

$$\left. \begin{aligned} z_1 &= \frac{\nu + \varrho - 2\nu\varrho + 2i\nu\varrho u}{\nu + \varrho + 2i\nu\varrho u} \\ z_2 &= \frac{2e^{i\varphi} \sqrt{\nu\varrho(1-\nu)(\varrho-1)}}{\nu + \varrho + 2i\nu\varrho u} \end{aligned} \right\} \quad (0 < \nu < 1 < \varrho, u \in \mathbb{R}, \varphi \in [0, 2\pi)) . \quad (5.5)$$

The hyperbolic-parabolic coordinate system is given by

$$\left. \begin{aligned} z_1 &= \frac{\nu + \varrho + 2i\nu\varrho u}{\nu + \varrho - 2\nu\varrho - 2i\nu\varrho u} \\ z_2 &= \frac{2ie^{i\varphi} \sqrt{\nu\varrho(1-\nu)(\varrho-1)}}{\nu + \varrho - 2\nu\varrho - 2i\nu\varrho u} \end{aligned} \right\} \quad (0 < \nu < 1 < \varrho, u \in \mathbb{R}, \varphi \in [0, 2\pi)) . \quad (5.6)$$

We introduce for the elliptic parabolic coordinate system [15, 18, 19] the new parameterisation  $\nu = 1/\cosh^2 \omega$  ( $\omega \in \mathbb{R}$ ), and  $\varrho = 1/\cos^2 \vartheta$ , ( $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$ ). In these coordinates we obtain for the Hamiltonian

$$\begin{aligned} \mathcal{H} = & \frac{1}{2m} \frac{\cos^2 \vartheta \cosh^2 \omega}{\cosh^2 \omega - \cos^2 \vartheta} \left[ p_\omega^2 + p_\vartheta^2 + (\coth^2 \omega + \cot^2 \vartheta) p_\alpha^2 \right. \\ & \left. + (\cosh^2 \omega \sinh^2 \omega + \sin^2 \vartheta \cos^2 \vartheta) p_u^2 + 2(\cosh^2 \omega - \cos^2 \vartheta) p_u p_\alpha \right] . \quad (5.7) \end{aligned}$$

The mixture of  $1/\cosh^2 \omega$ ,  $\cosh^2 \omega$  and  $\cosh^4 \omega$  makes it impossible to evaluate the path integral representation. The case for the hyperbolic-parabolic is similar and left to the reader.

### 5.3 The Semicircular-Parabolic Coordinate System

The semicircular-parabolic coordinate system is given by

$$\left. \begin{aligned} z_1 &= \frac{2\rho^2\nu^2u_1 - 2\rho\nu(\rho + \nu)u_2 - i[(\rho - \nu)^2 + \nu^2\rho^2(u_2^2 - 1)]}{2\rho^2\nu^2u_1 - 2\rho\nu(\rho + \nu)u_2 - i[(\rho - \nu)^2 + \nu^2\rho^2(u_2^2 + 1)]} \\ z_2 &= \frac{2\rho\nu u_2 - 2i(\rho + \nu)}{2\rho^2\nu^2u_1 - 2\rho\nu(\rho + \nu)u_2 - i[(\rho - \nu)^2 + \nu^2\rho^2(u_2^2 + 1)]} \end{aligned} \right\} \quad (\nu < 0 < \rho, u_1, u_2 \in \mathbb{R}) . \quad (5.8)$$

Redefining  $\rho = 2/\xi^2$  ( $\xi > 0$ ) and  $\nu^2 = -2/\eta^2$  ( $\eta > 0$ ) we find

$$\mathcal{H} = \frac{1}{2m} \frac{\xi^2\eta^2}{\xi^2 + \eta^2} \left[ p_\xi^2 + p_\eta^2 + (\xi^6 + \eta^6)p_{u_1}^2 + (\xi^2 + \eta^2)p_{u_2}^2 + 2(\xi^4 - \eta^4)p_{u_1}p_{u_2} \right] . \quad (5.9)$$

Although symmetric in  $\xi$  and  $\eta$  the involvement of quartic and sextic terms make any further evaluation impossible. There exist some attempts in the literature to treat such potential systems, and these studies go with the name “quasi-exactly solvable potentials” [44]. In fact, sextic oscillators with a centrifugal barrier and quartic hyperbolic and trigonometric can be considered, and they are very similar in their structure as for instance in (5.9). One can find particular solutions, provided the parameters in the quasi-exactly solvable potentials fulfil special conditions. Furthermore, well-defined expressions for the wave-functions and for the energy-spectrum can indeed be found if only quadratic, sextic, and a particular centrifugal term are present. The wave-functions then have the form of  $\Psi(x) \propto P(x^4) \times e^{-\alpha x^4}$ , with a polynomial  $P$ . However, quasi-exactly solvable potentials have the feature that only a *finite* number of bound states can be calculated (usually the ground state and some excited states). Another important observation is due to L  tourneau and Vinet [35]: They found quasi-exactly solvable potentials that emerge from dimensional reduction from two- and three-dimensional complex homogeneous spaces. The sextic potential in the Hamiltonian (5.9) is of that type. If we make a coordinate transformation from the “parabolic” coordinates  $(\xi, \eta)$  to the “Cartesian” coordinates  $(x, y)$  by means of  $x = \frac{1}{2}(\xi^2 - \eta^2)$ ,  $y = \xi\eta$  (5.9) is transformed into the Hamiltonian (4.14) with  $y = e^q$ , and nothing new can be obtained. Actually, the potential in (4.14) is called “Holt-potential” in  $\mathbb{R}^2$ , where it is maximally superintegrable, whereas its analogue in  $\mathbb{R}^3$  is minimally superintegrable [17]. Furthermore, analogues on the two-dimensional [18] and on the three-dimensional hyperboloid [19] exist which are separable in horicyclic and semi-circular parabolic coordinates, respectively, however only in horicyclic coordinates an analytic solution can be found. Therefore, we are not able to treat systems with the structure of (5.9) any further.

## 6 Superintegrable Potentials on the Two-Dimensional Hyperboloid

As has been pointed out by Kalnins, Miller, Hakobyan, and Pogosyan [30] the properties of the quantum motion on  $\text{HH}(2)$  have as a by-result that two potentials emerge which are superintegrable on the two-dimensional hyperboloid. These two potentials are

$$V_1 = \frac{\hbar^2}{2m} \left( \frac{\alpha^2 - \frac{1}{4}}{u_2^2} - \frac{\gamma^2}{(u_0 - u_1)^2} + \beta^2 \frac{u_0 + u_1}{(u_0 - u_1)^3} \right) \left\{ \begin{array}{l} \text{equidistant} \\ \text{elliptic-parabolic} \\ \text{hyperbolic-parabolic} \\ \text{horicyclic} \end{array} \right\} , \quad (6.1)$$

$$V_2 = \frac{\hbar^2}{2m} \left( \frac{\alpha^2 - \frac{1}{4}}{u_2^2} + \gamma^2 \frac{u_0 u_1}{(u_0^2 + u_1^2)^2} + (\alpha^2 - \beta^2) \frac{u_0^2 - u_1^2}{(u_0^2 + u_1^2)^2} \right) \left\{ \begin{array}{l} \text{equidistant} \\ \text{semi-hyperbolic} \end{array} \right\}. \quad (6.2)$$

The two-dimensional hyperboloid is characterised by  $u_0^2 - u_1^2 - u_2^2 = 1$  with  $u_0 > 0$ . In (6.1) and (6.2) we have listed on the right hand side the coordinate systems which allowed separation of variables in the Schrödinger equation and the path integral. The underlined coordinate systems allow a complete path integral treatment of the potential in question. In elliptic-parabolic, hyperbolic-parabolic and semi-hyperbolic coordinates no explicit path integral solution is possible. In [30] explicit solutions for the two potential in all the separable coordinate systems were given in terms of power expansions (polynomials) in the respective coordinates, including interbasis expansions which relate one solution to another. However, only the bound states solutions were given. The equidistant coordinate system is given by

$$u_0 = \cosh \tau_1 \cosh \tau_2, \quad u_1 = \cosh \tau_1 \sinh \tau_2, \quad u_2 = \sinh \tau_1, \quad (\tau_1, \tau_2 \in \mathbb{R}), \quad (6.3)$$

and the horicyclic system has the form

$$u_0 = \frac{y^2 + x^2 + 1}{2y}, \quad u_1 = \frac{y^2 + x^2 - 1}{2y}, \quad u_2 = \frac{x}{y}, \quad (y > 0, x \in \mathbb{R}). \quad (6.4)$$

We consider the potential  $V_1$  in equidistant and horicyclic and the potential  $V_2$  in equidistant coordinates.

(Equidistant coordinates:)

$$V_1 = \frac{\hbar^2}{2m} \left[ \frac{\alpha^2 - \frac{1}{4}}{\sinh^2 \tau_1} + \frac{1}{\cosh^2 \tau_1} (\beta^2 e^{4\tau_2} - \gamma^2 e^{2\tau_2}) \right], \quad (6.5)$$

(Horicyclic coordinates:)

$$= \frac{\hbar^2}{2m} y^2 \left( \frac{\alpha^2 - \frac{1}{4}}{x^2} - \gamma^2 + \beta^2 x^2 + \beta^2 y^2 \right), \quad (6.6)$$

(Equidistant coordinates:)

$$V_2 = \frac{\hbar^2}{2m} \left[ \frac{\alpha^2 - \frac{1}{4}}{\sinh^2 \tau_1} + \frac{1}{\cosh^2 \tau_1} \left( \frac{\alpha^2 - \beta^2}{\cosh^2 2\tau_2} - \frac{\gamma}{2 \cosh^2 2\tau_2} \right) \right]. \quad (6.7)$$

With our results from sections 3 and 4 we can evaluate the corresponding path integrals. We see that the path integral for the potential  $V_1$  in equidistant coordinates corresponds to the path integral (3.45), the path integral for the potential  $V_1$  in horicyclic coordinates to the path integral (4.7), and the path integral for  $V_2$  in equidistant in equidistant coordinates to the path integral (3.27)<sup>1</sup>. The principal difference between the set of the potentials  $V_1$  and  $V_2$  and the path integrals (3.45, 4.7, 3.27) is that we can choose freely the constants in the potentials, whereas in the path integrals (3.45, 4.7, 3.27) the couplings in the potentials are fixed and related to each other. This interrelation of the constants in the path integrals (3.45, 4.7, 3.27) has on the one hand side the consequence that the principal spectrum on  $\text{HH}(2)$  is always continuous with form (2.22). This freedom of the choice of the couplings  $\alpha, \beta, \gamma$ , and in particular  $\gamma$  (it lowers the

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<sup>1</sup>Actually, the path integral formulations for the potentials  $V_1$  and  $V_2$  in the other coordinate systems elliptic- and hyperbolic-parabolic, and semi-hyperbolic coordinates correspond to the path integral formulations to the corresponding coordinate systems on  $\text{HH}(2)$  which we do not state explicitly.

potential trough due to its sign), has on the other hand the consequence that for the principal quantum number corresponding to, say,  $\tau_1$  or  $y$ , also a finite number of discrete states are allowed with the maximal number in  $V_1$  given by  $N_{\max} < [\frac{1}{2}(\gamma^2/2\beta - \alpha - 1)]$ . Therefore, in order to perform a path integral evaluation for  $V_1$  and  $V_2$  we have to take the path integrals (3.45, 4.7, 3.27) and replace the couplings accordingly. The scattering states for  $V_1$  and  $V_2$  follows immediately from our solutions. The discrete solutions can be obtained from the discrete spectrum of the modified Pöschl–Teller potential and the Morse potential, respectively, by inserting the couplings accordingly. In fact, only the bound state solutions of the sub-path integration give bound state solutions corresponding to the principal quantum number corresponding to, say,  $\tau_1$  or  $y$ . Because these bound state solutions have been presented in [30] in great detail, this will not be repeated here.

## 7 Discussion and Conclusion

In this paper we have successfully evaluated the path integral on the Hermitian space  $\text{HH}(2)$  by six coordinate variable out of twelve which separate the Schrödinger equation and the path integral formulation. In each case we could separate off the ignorable coordinates by a two-dimensional Gaussian path integration. The remaining problems had the structure of a path integral on the two-dimensional hyperboloid equipped with a potential. There occurred (modified) Pöschl–Teller potentials, a barrier potential, the Morse potential, and the (radial) harmonic oscillator. In some cases a part of the solutions contained in a sub-path integration (sub-group decomposition) a discrete and a continuous spectrum. However, the principal spectrum is always continuous and has the form

$$E_p = \frac{\hbar^2}{2m}(p^2 + 4) . \quad (7.1)$$

The zero-energy  $E_0 = 2\hbar^2/m$  is a well-known feature of the quantum motion on a space of constant negative curvature.

We summarise the results in the Table 1. We have omitted the ignorable coordinates because they just give exponentials, and the term “Legendre functions” is used synonymously with “hypergeometric functions”. In the three parametric (two elliptic and the semi-hyperbolic) and in the three parabolic coordinate systems no solution could be found. In the case of the elliptic systems this is due to our ignorance of a theory of special functions in terms of such coordinates, and in the case of the three parabolic coordinates solutions could not be found due to the high anharmonicity of the emerging potential problems.

We have observed that a free motion in some space (here with non-constant curvature, though constant sectional curvature) leads to potential-coupling after integrating out the ignorable coordinates, i.e. to interaction. This feature has been pointed out in [2]. Due to the structure of  $F(\mathbf{x}, \mathbf{y})$  we also see that the metric is  $(-, +, \dots, +)$ , i.e. it is of the Minkowski-type, and hence the Hamiltonian system under consideration is integrable and relativistic with non-trivial interaction. Choosing different coordinate systems yields different potential-interactions which are, however, all equivalent in the sense of quantum motion in  $\text{HH}(2)$ . Some were also identical and yield superintegrable potentials on the two-dimensional hyperboloid. The emerging of interaction after separating off ignorable coordinates of the free motion in an homogeneous space, is of course not restricted to the space  $\text{HH}(2)$ . In fact, also the path integral formulations of the Pöschl–Teller potential is due to path integration on the homogeneous space corresponding to  $\text{SU}(2)$  [1, 6, 33] and the path integral formulations of the modified Pöschl–Teller potential is due

Table 1: Solutions of the path integration in Hermitian Space HH(2)

| Coordinate system | Solution in terms of the wave-functions        | Potentials                                     |
|-------------------|--|--|
| Spherical         | Product of Legendre functions                  | Pöschl–Teller and modified Pöschl–Teller       |
| Equidistant-I     | Product of Legendre functions                  | modified Pöschl–Teller                         |
| Equidistant-II    | Product of Legendre functions                  | Hyperbolic barrier and modified Pöschl–Teller  |
| Equidistant-III   | W-Whittaker function times Legendre function   | Morse potential and modified Pöschl–Teller     |
| Horicyclic-I      | Laguerre polynomial times W-Whittaker function | Radial harmonic oscillator and Morse potential |
| Horicyclic-II     | Hermite polynomial times W-Whittaker function  | Harmonic oscillator and Morse potential        |

to path integration on the homogeneous space corresponding to  $SU(1, 1)$  [1, 8, 33]. The same is true for the radial harmonic oscillator, and an analogous consideration was done in [15, 16, 21] for a path integral identity involving spheroidal coordinates. The latter are in fact examples of the quantum motion in three-dimensional flat space  $\mathbb{R}^3$  and on the three-dimensional sphere.

We have therefore also shown that the path integral solutions on HH(2) gives path integral identities for potential problems, a property which is valid for every solution after performing the Gaussian path integration of the ignorable coordinates. In particular, it turns out that two such potentials, denoted by  $V_1$  and  $V_2$  are superintegrable potentials on the two-dimensional hyperboloid. The evaluation of the bound state solutions have been achieved for  $V_1$  and  $V_2$  in [30], whereas our contribution yields also the scattering states.

The Hermitian hyperbolic space is closely related to the case of the quantum motion in hyperbolic spaces of rank one. A path integral discussion was performed in [11], however restricted to a particular coordinate system only. In the space  $SU(n, 1)/S[U(1) \times U(n)]$  we have for the metric

$$ds^2 = \frac{dy^2}{y^2} + \frac{1}{y^2} \sum_{k=2}^n dz_k dz_k^* + \frac{1}{y^4} \left( dx_1 + \Im \sum_{k=2}^n z_k^* dz_k \right)^2, \quad (7.2)$$

( $z_k = x_k + iy_k \in \mathbb{C}$  ( $k = 2, \dots, n$ ),  $x_1 \in \mathbb{R}$ ,  $y > 0$ ), with the hyperbolic distance given by

$$\cosh d(\mathbf{q}'', \mathbf{q}') = \frac{((\mathbf{x}'' - \mathbf{x}')^2 + y'^2 + y''^2)^2 + 4(x_1'' - x_1' + (\mathbf{x}'' \mathbf{y}' - \mathbf{y}'' \mathbf{x}'))^2}{4(y' y'')^4}. \quad (7.3)$$

If we additionally introduce a set of polar coordinates, this space is a  $n$ -dimensional generalisation of HH(2) in terms of horicyclic-I coordinates  $z_k = r_k e^{i\varphi_k}$ , ( $r_k > 0$ ,  $0 \leq \varphi_k \leq 2\pi$ ,  $k = 2, \dots, n$ ). If we set  $n = 2$ , we recover the present case of HH(2). It is obvious that the higher the dimension the more separable coordinate systems can be found. As mentioned in [2] the case of HH(2) is rather special because all separable coordinate systems have *exactly* two ignorable and two non-ignorable coordinates. This is due to the property of  $SU(2, 1)$  has four mutually non-conjugate maximal Abelian subgroups which are all two-dimensional. In [4] separable coordinate systems



on general Hermitian hyperbolic spaces were considered with the number of ignorable coordinates equals to  $n = p + q - 1$ . For the higher dimensional case we have thus a Hermitian hyperbolic space  $\text{HH}(3)$  with three ignorable coordinates and three non-ignorable coordinates, the coordinates on the three-dimensional hyperboloid. In the latter there are 34 of such systems which separate the Helmholtz, respectively the Schrödinger equation, and the path integral. Following [30] we can identify superintegrable potentials on the three-dimensional hyperboloid.

One could also find a similar line of reasoning in [4] where the case of motion on the corresponding  $\text{SU}(2, 2)$ -hyperboloid was worked out. Here, the corresponding reduced space of the non-ignorable coordinates is the  $\text{O}(2, 2)$ -hyperboloid, where 75 coordinate systems could be identified [29], and 11 different types of superintegrable potentials. These potentials were stated, but exact solutions of the corresponding Schrödinger equation were not worked out.

It would be also desirable to obtain a closed expression of the Green's function  $G(\cosh d; E)$  on  $\text{HH}(2)$  (respectively on  $\text{HH}(n)$ ) in terms of  $\cosh d$ . Studies along these lines will be subject to future investigations.

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## A Formulation of the Path Integral in Curved Spaces

In order to set up our notation for path integrals on curved manifolds I proceed in a canonical way. To avoid unnecessary overlap with our Table of Path Integrals [21] I give in the following only the essential information required for the path integral representation on curved spaces. For more details concerning ordering prescriptions, transformation techniques, perturbation expansions, point interactions, and boundary conditions I refer to [21], where also listings of the application of Basic Path Integrals will be presented. In the following  $\mathbf{q}$  denote some  $D$ -dimensional coordinates. I start by considering the classical Lagrangian corresponding to the line element  $ds^2 = g_{ab}dq^a dq^b$  of the classical motion in some  $D$ -dimensional Riemannian space

$$\mathcal{L}_{Cl}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2} \left( \frac{ds}{dt} \right)^2 - V(\mathbf{q}) = \frac{m}{2} g_{ab}(\mathbf{q}) \dot{q}^a \dot{q}^b - V(\mathbf{q}) . \quad (\text{A.1})$$

The quantum Hamiltonian is *constructed* by means of the Laplace-Beltrami operator

$$H = -\frac{\hbar^2}{2m} \Delta_{LB} + V(\mathbf{q}) = -\frac{\hbar^2}{2m} \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^a} g^{ab} \sqrt{g} \frac{\partial}{\partial q^b} + V(\mathbf{q}) \quad (\text{A.2})$$

as a *definition* of the quantum theory on a curved space. Here are  $g = \det(g_{ab})$  and  $(g^{ab}) = (g_{ab})^{-1}$ . The scalar product for wavefunctions on the manifold reads  $(f, g) = \int d\mathbf{q} \sqrt{g} f^*(\mathbf{q}) g(\mathbf{q})$ , and the momentum operators which are hermitian with respect to this scalar product are given by

$$p_a = \frac{\hbar}{i} \left( \frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2} \right) , \quad \Gamma_a = \frac{\partial \ln \sqrt{g}}{\partial q^a} . \quad (\text{A.3})$$

In terms of the momentum operators (A.3) we can rewrite  $\mathbb{H}$  by using a product according to  $g_{ab} = h_{ac}h_{cb}$  [21]. Then we obtain for the Hamiltonian (A.2) (PF - Product-Form)

$$\mathbb{H} = -\frac{\hbar^2}{2m}\Delta_{LB} + V(\mathbf{q}) = \frac{1}{2m}h^{ac}p_ap_bh^{cb} + \Delta V_{PF}(\mathbf{q}) + V(\mathbf{q}) , \quad (\text{A.4})$$

and for the path integral

$$\begin{aligned} & K(\mathbf{q}'', \mathbf{q}'; T) \\ &= \int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''} \mathcal{D}_{PF}\mathbf{q}(t) \sqrt{g(\mathbf{q})} \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} h_{ac}(\mathbf{q}) h_{cb}(\mathbf{q}) \dot{q}^a \dot{q}^b - V(\mathbf{q}) - \Delta V_{PF}(\mathbf{q}) \right] dt \right\} \\ &= \lim_{N \rightarrow \infty} \left( \frac{m}{2\pi i \epsilon \hbar} \right)^{ND/2} \prod_{k=1}^{N-1} \int d\mathbf{q}_k \sqrt{g(\mathbf{q}_k)} \\ &\quad \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^N \left[ \frac{m}{2\epsilon} h_{bc}(\mathbf{q}_j) h_{ac}(\mathbf{q}_{j-1}) \Delta q_j^a \Delta q_j^b - \epsilon V(\mathbf{q}_j) - \epsilon \Delta V_{PF}(\mathbf{q}_j) \right] \right\} . \end{aligned} \quad (\text{A.5})$$

$\Delta V_{PF}$  denotes the well-defined quantum potential

$$\Delta V_{PF}(\mathbf{q}) = \frac{\hbar^2}{8m} \left[ g^{ab} \Gamma_a \Gamma_b + 2(g^{ab} \Gamma_b)_{,b} + g^{ab}_{,ab} \right] + \frac{\hbar^2}{8m} \left( 2h^{ac} h^{bc}_{,ab} - h^{ac}_{,a} h^{bc}_{,b} - h^{ac}_{,b} h^{bc}_{,a} \right) \quad (\text{A.6})$$

arising from the specific lattice formulation (A.5) of the path integral or the ordering prescription for position and momentum operators in the quantum Hamiltonian, respectively. Here we have used the abbreviations  $\epsilon = (t'' - t')/N \equiv T/N$ ,  $\Delta \mathbf{q}_j = \mathbf{q}_j - \mathbf{q}_{j-1}$ ,  $\mathbf{q}_j = \mathbf{q}(t' + j\epsilon)$  ( $t_j = t' + \epsilon j$ ,  $j = 0, \dots, N$ ) and we interpret the limit  $N \rightarrow \infty$  as equivalent to  $\epsilon \rightarrow 0$ ,  $T$  fixed. The lattice representation can be obtained by exploiting the composition law of the time-evolution operator  $U = \exp(-iHT/\hbar)$ , respectively its semi-group property.

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